ELSEVIER Journal of Pure and Applied Algebra 129 (1998) 101–109

JOURNAL OF PURE AND APPLIED ALGEBRA

The quantum Cayley–Hamilton theorem[1]

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Communicated by J.D. Stasheff; received 21 June 1993; received in revised form 15 November 1996

Abstract

Let $X=(x_{ij})_{n\times n}$ be the generic matrix of the quantum group $K[GL_q(n)]$. First we prove that X satisfies two quantum characteristic equations, both become the classical characteristic equation when $q=1$. Second we prove a quantum version of Muir's formula for X. © Elsevier Science B.V. All rights reserved.

AMS Classification: 15A24, 17B37

1. Definitions and basic properties

Let K be a field and let q be a nonzero element in K. The quantum group $K[GL_q(n)]$ is a deformation of the Hopf algebra $K[GL(n)]$; as an algebra, it is defined by

$$K[GL_q(n)]=K[M_q(n)][D^{-1}] \quad \text{and} \quad K[M_q(n)]=K\langle x_{ij}\rangle/(R_q)$$

where $K\langle x_{ij}\rangle$ is the free algebra generated by $\{x_{ij}\,|\,i,j=1,\dots,n\}$ and (R_q) is the ideal generated by the following quadratic relations:

$$\begin{aligned}
&x_{jt}x_{it}=qx_{it}x_{jt} && \text{for all } i<j \text{ and } t,\\
&x_{it}x_{is}=qx_{is}x_{it} && \text{for all } s<t \text{ and } i,\\
&x_{it}x_{js}=x_{js}x_{it} && \text{for all } i<j \text{ and } s<t,\\
&x_{is}x_{jt}-x_{jt}x_{is}=(q-q^{-1})x_{it}x_{js} && \text{for all } i<j \text{ and } s<t.
\end{aligned}$$

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[1] The results of this paper were presented at the AMS summer research institute at Pennsylvania State University, University Park, 8–16 July 1991. The author wishes to thank Professors I. Frenkel, E. Friedlander, W. Haboush, J. Jantzen and B. Parshall for organizing the wonderful conference.

PII S0022-4049(97)00039-X

The relation vector space R_q has dimension $\frac{1}{2}n^2(n^2-1)$ as in the classical case $(q=1)$. The basic properties of $K[GL_q(n)]$ can be found in [1]. The quantum determinant, denoted by D, is

$$D=\sum_{\sigma\in S_n}(-q)^{-l(\sigma)}x_{\sigma(1)1}\cdots x_{\sigma(n)n}=\sum_{\sigma\in S_n}(-q)^{-l(\sigma)}x_{1\sigma(1)}\cdots x_{n\sigma(n)}$$

where S_n is the nth symmetric group, $l(i_1,\ldots,i_n)$ is the number of inversions of the sequence $(i_1,\ldots,i_n)$ and $l(\sigma)=l(\sigma(1),\ldots,\sigma(n))$. The coalgebra structure of $K[M_q(n)]$ and $K[GL_q(n)]$ is determined by the following rules:

$$\Delta(x_{ij})=\sum_{k=1}^{n}x_{ik}\otimes x_{kj}\quad\text{for all } i \text{ and } j,$$

$$\varepsilon(x_{ij})=\delta_{ij}\qquad\text{for all } i \text{ and } j,$$

where δ_{ij} is the Kronecker delta. The matrix $X=(x_{ij})_{n\times n}$ is called the generic matrix of $K[M_q(n)]$ and $K[GL_q(n)]$. If $q\neq 1$, the entries of X are noncommutative.

The left and right quantum minors (or sub-determinants) and left and right quantum sub-permanents play a key role in this paper. Let $(i_1,\ldots,i_m)$ and $(j_1,\ldots j_m)$ be two sequences of integers between 1 and n. Here neither $(i_1,\ldots,i_m)$ nor $(j_1,\ldots,j_m)$ needs to be increasing. The left and right quantum minors $D_{\mathrm{l}}(i_s|j_s)$ and $D_{\mathrm{r}}(i_s|j_s)$ are

$$D_{\mathrm{l}}(i_s|j_s)=\sum_{\sigma\in S_m}(-q)^{-l(j_{\sigma(s)})+l(j_s)}x_{i_1j_{\sigma(1)}}\cdots x_{i_mj_{\sigma(m)}},$$

$$D_{\mathrm{r}}(i_s|j_s)=\sum_{\sigma\in S_m}(-q)^{-l(i_{\sigma(s)})+l(i_s)}x_{i_{\sigma(1)}j_1}\cdots x_{i_{\sigma(m)}j_m}.$$

The left and right quantum sub-permanents $P_{\mathrm{l}}(i_s|j_s)$ and $P_{\mathrm{r}}(i_s|j_s)$ are

$$P_{\mathrm{l}}(i_s|j_s)=\sum_{\sigma\in S_m}q^{l(j_{\sigma(s)})-l(j_s)}x_{i_1j_{\sigma(1)}}\cdots x_{i_mj_{\sigma(m)}},$$

$$P_{\mathrm{r}}(i_s|j_s)=\sum_{\sigma\in S_m}q^{l(i_{\sigma(s)})-l(i_s)}x_{i_{\sigma(1)}j_1}\cdots x_{i_{\sigma(m)}j_m}.$$

The following properties of quantum minors and quantum sub-permanents can be found in [1], which can be proved easily by using corepresentations of $K[M_q(n)]$.

(P1.1) Given a positive integer $m\leq n$, let Φ_m denote the set

$$\Phi_m=\{(i_1,\ldots,i_m)|\,1\leq i_1<\cdots<i_m\leq n\}.$$

If (i_s) and (j_s) are in Φ_m, then $D_{\mathrm{l}}(i_s|j_s)=D_{\mathrm{r}}(i_s|j_s)$; in this case, it is called quantum minor and denoted by $D(i_s|j_s)$. Similarly $P_{\mathrm{l}}(i_s|j_s)=P_{\mathrm{r}}(i_s|j_s)$, and it is called quantum sub-permanent and denoted by $P(i_s|j_s)$. The set Φ_0 has only one element which is the empty set ϕ. We assume that $P(\phi|\phi)=D(\phi|\phi)=1$ (used in Section 3). By definition, $P(i|j)=D(i|j)=x_{ij}$ and $D(1\cdots n|1\cdots n)=D$. The sub-permanent $P(1\cdots n|1\cdots n)$ is denoted by P and it is called the quantum permanent of X. In general, $P(i_s|j_s) \neq D(i_s|j_s)$ when $m \geq 2$.

(P1.2) A left (resp. right) quantum minor $D_l(i_s|j_s)$ (resp. $D_r(i_s|j_s)$) is equal to zero if and only if either $i_s = i_t$ for some $s < t$ or $j_s = j_t$ for some $s < t$. This statement is not true for the quantum sub-permanents. If (i'_s) (resp. (j'_s)) is a permutation of a nondecreasing sequence (i_s) (resp. (j_s)), then

$$D_l(i'_s|j'_s) = (-q)^{l(j'_s)-l(i'_s)} D(i_s|j_s) \quad \text{and} \quad D_r(i'_s|j'_s) = (-q)^{l(i'_s)-l(j'_s)} D(i_s|j_s)$$

and

$$P_l(i'_s|j'_s) = q^{l(i'_s)-l(j'_s)} P(i_s|j_s) \quad \text{and} \quad P_r(i'_s|j'_s) = q^{l(j'_s)-l(i'_s)} P(i_s|j_s).$$

(P1.3) Quantum versions of the Laplace expansion hold:

$$D_l(i_1 \cdots i_m | j_1 \cdots j_m) = \sum_{\sigma \in S_m^k} (-q)^{-l(j_{\sigma(s)})+l(j_s)} D_l(i_1 \cdots i_k | j_{\sigma(1)} \cdots j_{\sigma(k)}) D_l(i_{k+1} \cdots i_m | j_{\sigma(k+1)} \cdots j_{\sigma(m)}),$$

$$D_r(i_1 \cdots i_m | j_1 \cdots j_m) = \sum_{\sigma \in S_m^k} (-q)^{-l(i_{\sigma(s)})+l(i_s)} D_r(i_{\sigma(1)} \cdots i_{\sigma(k)} | j_1 \cdots j_k) D_r(i_{\sigma(k+1)} \cdots i_{\sigma(m)} | j_{k+1} \cdots j_m),$$

$$P_l(i_1 \cdots i_m | j_1 \cdots j_m) = \sum_{\sigma \in S_m^k} q^{l(j_{\sigma(s)})-l(j_s)} P_l(i_1 \cdots i_k | j_{\sigma(1)} \cdots j_{\sigma(k)}) P_l(i_{k+1} \cdots i_m | j_{\sigma(k+1)} \cdots j_{\sigma(m)}),$$

$$P_r(i_1 \cdots i_m | j_1 \cdots j_m) = \sum_{\sigma \in S_m^k} q^{l(i_{\sigma(s)})-l(i_s)} P_r(i_{\sigma(1)} \cdots i_{\sigma(k)} | j_1 \cdots j_k) P_r(i_{\sigma(k+1)} \cdots i_{\sigma(m)} | j_{k+1} \cdots j_m),$$

where $S_m^k = \{\sigma \in S_m | \sigma(1) < \cdots < \sigma(k) \text{ and } \sigma(k+1) < \cdots < \sigma(m)\}$ consists of all k-shuffles in S_n.

2. The Cayley–Hamilton theorem

Let $A = (a_{ij})_{n \times n}$ be an $n \times n$ matrix over a commutative ring. Then A satisfies the classical characteristic equation

$$A^n - tr_1 A^{n-1} + \cdots + (-1)^{n-1} tr_{n-1} A + (-1)^n \det A I_{n \times n} = 0$$

where the kth trace tr_k is the sum of all $k \times k$ principal minors of A, $\det A$ is the determinant of A and $I_{n \times n}$ is the $n \times n$ identity matrix.

It is easy to check that if $q \neq 1$, the generic matrix X of the quantum group $K[GL_q(n)]$ does not satisfy the classical characteristic equation. The aim of this section is to prove X satisfies two quantum versions of the characteristic equation. We need to define quantum trace and other notations. For simplicity, let $q_{ij} = q$ if $i < j$, $q_{ij} = 1$ if $i = j$, and $q_{ij} = q^{-1}$ if $i > j$. For every i, j and m, denote

$$tr_m(j) = \sum_{(i_s) \in \Phi_m} \left(\prod_{t=1}^{m} q_{ji_t} \right) D(i_s | i_s),$$

$$Tr_m = (tr_m(j)\delta_{ij})_{n \times n},$$

$$b_m(ij) = \sum_{(i_s) \in \Phi_m} D_1(i, (i_s) | (i_s), j).$$

So Tr_m is a diagonal $n \times n$-matrix. Let B_m denote the $n \times n$-matrix $(b_m(ij))_{n \times n}$. The following lemma gives some relations between X, B_m and Tr_m.

Lemma 2.1. (1) $B_0 = X$.

(2) $B_{n-1} = (-1)^{n-1} Tr_n$ *and* $tr_n(i) = q^{n+1-2i} D$.

(3) $B_m = 0$ *for all* $m \geq n$.

(4) $B_m = (-1)^m X \cdot Tr_m + X \cdot B_{m-1}$ *for all* $m = 1, \ldots, n-1$.

(5) $B_m = \sum_{k=0}^{m} (-1)^{m-k} X^{k+1} \cdot Tr_{m-k}$ *for all* $m = 1, \ldots, n-1$.

Proof. (1) If $m = 0$, $b_0(ij) = D_1(i|j) = x_{ij}$, and $B_0 = X$.

(2) By (P1.2), $b_{n-1}(i,j) = 0$ for all $i \neq j$ and $b_{n-1}(ii) = (-1)^{n-1} tr_n(i)$, which gives that $B_{n-1} = (-1)^{n-1} Tr_n$. An easy computation shows that $tr_n(i) = q^{n+1-2i} D$.

(3) By (P1.2), for all $m \geq n$, $b_m(ij) = 0$ and then $B_m = 0$.

(4) By (P1.2) and (P1.3),

$$\begin{aligned} D_1(i, (i_s) | (i_s), j) &= \prod_{t=1}^{m} (-q_{ji_t}) x_{ij} D_1(i_s | i_s) \\ &\quad + \sum_{t=1}^{m} x_{ii_t} D_1(i_t i_1 \cdots i_{t-1} \hat{i}_t i_{t+1} \cdots i_m | i_1 \cdots i_{t-1} \hat{i}_t i_{t+1} \cdots i_m, j), \end{aligned}$$

and hence

$$\begin{aligned} b_m(ij) &= \sum_{(i_s) \in \Phi_m} D_1(i, (i_s) | (i_s), j) \\ &= (-1)^m x_{ij} \left(\sum_{(i_s) \in \Phi_m} \prod_{t=1}^{m} q_{ji_t} D(i_s | i_s) \right) + \sum_{i_s=1}^{n} x_{ii_s} b_{m-1}(i_s j). \end{aligned}$$

Therefore we obtain $B_m = (-1)^m X \cdot Tr_m + X B_{m-1}$.

(5) This follows from (4) and straightforward induction. □

We are ready to prove the first quantum version of the characteristic equation.

Theorem 2.2. *The generic matrix $X=(x_{ij})_{n\times n}$ satisfies the following characteristic equation*:

$$X^n - X^{n-1}\cdot Tr_1 + \cdots + (-1)^{n-1}X\cdot Tr_{n-1} + (-1)^n Tr_n = 0.$$

Proof. By Lemma 2.1(5),

$$LHS = \left(\sum_{k=0}^{n-1}(-1)^{n-1-k}X^{k+1}\, Tr_{n-1-k}\right) + (-1)^n Tr_n = B_{n-1} - B_{n-1} = 0. \quad \square$$

By definition, only when $q=1$, Tr_k become the classical kth trace function of X. In Theorem 2.2, we use the ordinary powers of X and quantum traces Tr_k. Next we will prove another quantum version of characteristic equation by using ordinary trace functions tr_k and quantum powers of X. Let $Y=(y_{ij})_{n\times n}$ be an $n\times n$ matrix over $K[GL_q(n)]$. The q-multiplication of X and Y is defined by

$${}_q(X\cdot Y) = ({}_q(X\cdot Y)_{ij})_{n\times n} \quad \text{and} \quad {}_q(X\cdot Y)_{ij} = \sum_{k=1}^{n} x_{ik}q_{kj}y_{kj}.$$

It is easy to see that the q-multiplication is generally not associative. The mth q-power of X is defined as follows:

$${}_qX^1 = X \quad \text{and} \quad {}_qX^{m+1} = {}_q(X\cdot {}_qX^m).$$

As before, we need some notations:

$$tr_m = \sum_{(i_s)\in\Phi_m} D(i_s|i_s),$$

$$C_m(ij) = (-1)^m \sum_{(i_s)\in\Phi_m} D_1(i,(i_s)|j,(i_s)),$$

$$C_m = (c_m(ij))_{n\times n}.$$

The relations between ${}_qX^m$, C_m and tr_m are the following.

Lemma 2.3. (1) $C_0 = X$.

(2) $C_{n-1} = (-1)^{n-1}tr_n I_{n\times n} = (-1)^{n-1}DI_{n\times n}$.

(3) $C_m = 0$ *for all* $m\geq n$.

(4) $C_m = (-1)^m X tr_m + {}_q(X\cdot C_{m-1})$ *for all* $m=1,\ldots,n-1$.

(5) $C_m = \sum_{k=0}^{m}(-1)^{m-k}{}_qX^{k+1}tr_{m-k}$ *for all* $m=1,\ldots,n-1$.

Proof. (1)–(3), and (5) can be proved in the same way as those in Lemma 2.1 (see the proof of Lemma 2.1).

(4) By (P1.2) and (P1.3),

$$D_1(i,(i_s)|j,(i_s)) = x_{ij}D(i_s|i_s) + \sum_{t=1}^{m}(-q_{i_tj})x_{ii_t}D_1(i_ti_1\cdots i_{t-1}\hat{i}_ti_{t+1}\cdots i_m|j,i_1\cdots i_{t-1}\hat{i}_ti_{t+1}\cdots i_m),$$

and hence

$$\begin{aligned} c_m(ij) &= (-1)^m \sum_{(i_s)\in \Phi_m} D_1(i,(i_s)|j,(i_s)) \\ &= (-1)^m x_{ij} \sum_{(i_s)\in \Phi_m} D(i_s|i_s) + (-1)^{m-1} \sum_{i_t=1}^{n} x_{ii_t} q_{i_t j} c_{m-1}(i_t j). \end{aligned}$$

Therefore it follows that $C_m = (-1)^m X\, tr_m + {}_q(X\, C_{m-1})$. □

Now we are ready to prove the second quantum version of the characteristic equation.

Theorem 2.4. *The generic matrix* $X=(x_{ij})_{n\times n}$ *satisfies the following characteristic equation*:

$${}_qX^n - {}_qX^{n-1} tr_1 + \cdots + (-1)^{n-1} X\, tr_{n-1} + (-1)^n D I_{n\times n} = 0.$$

Proof. By Lemma 2.3(5)

$$LHS = \left(\sum_{k=0}^{n-1} (-1)^{n-1-k} {}_qX^{k+1}\, tr_{n-1-k} \right) + (-1)^n D I_{n\times n} = C_{n-1} - C_{n-1} = 0. \qquad \square$$

By definition, ${}_qX^m = X^m$ when $q=1$. Hence, if $q=1$, both quantum characteristic equations reduce to the classical one.

3. Muir's formula

Let $A=(a_{ij})_{n\times n}$ be an $n\times n$ matrix over a commutative ring. Muir's formula states that

$$\sum_{m=0}^{n} (-1)^m \sum_{(i_s)\in \Phi_m} P(i_s|i_s) D(1\cdots(\hat{i_s})\cdots n|1\cdots(\hat{i_s})\cdots n) = 0.$$

Here $(1\cdots(\hat{i_s})\cdots n)$ is the complement of $(i_1\cdots i_m)$ in the set $(1,\ldots,n)$. As in Section 2, $P(i_s|i_s)$ (resp. $D(1\cdots(\hat{i_s})\cdots n|1\cdots(\hat{i_s})\cdots n)$) is an $m\times m$ principal sub-permanent (resp. $(n-m)\times(n-m)$ principal minor) of A. In this section we will prove Muir's formula for the generic matrix X of the quantum group $K[GL_q(n)]$. As usual we need some new notations. Denote

$$\begin{aligned} O_m = \sum_{(i_s)\in \Phi_m, i_1>1} \Bigg\{ & P(i_s|i_s) D(1\cdots(\hat{i_s})\cdots n|1\cdots(\hat{i_s})\cdots n) \\ & + \Bigg[\sum_{t=1}^{m} q^{-1} P_r(i_1\cdots \hat{i}_t \cdots i_m, 1|i_1\cdots \hat{i}_t\cdots i_m i_t) \end{aligned}$$

$$\times\ D_l(i_t 2\cdots(\hat{i_s})\cdots n|1,2\cdots(\hat{i_s})\cdots n)\Big]\Big]\Big\},$$

$$E_m = \sum_{(i_s)\in\Phi_m} P(i_s|i_s)D(1\cdots(\hat{i_s})\cdots n|1\cdots(\hat{i_s})\cdots n).$$

The relations between O_m and E_m are the following:

Lemma 3.1. (1) $E_1 - O_1 = D = D(1\cdots n|1\cdots n)$.
(2) $O_{n-1} = P = P(1\cdots n|1\cdots n)$.
(3) $O_m = E_{m+1} - O_{m+1}$ *for all* $m = 1,\cdots,n-2$.

Proof. We will use (P1.2) and (P1.3) and the following versions of the Laplace expansion:

$$D_l(i,(i_s)|j,(i_s)) = x_{ij}D(i_s|i_s) + \sum_{t=1}^{m}(-q_{i_t j})x_{ii_t}D_l(i_t i_1\cdots i_{t-1}\hat{i}_t i_{t+1}\cdots i_m|j,i_1\cdots i_{t-1}\hat{i}_t i_{t+1}\cdots i_m),$$

$$P_r((i_s),i|(i_s),j) = P(i_s|i_s)x_{ij} + \sum_{t=1}^{m} q_{i_t i}P_r(i_1\cdots i_{t-1}\hat{i}_t i_{t+1}\cdots i_m,i|i_1\cdots i_{t-1}\hat{i}_t i_{t+1}\cdots i_m i_t)x_{i_t j}.$$

(1)

$$\begin{aligned} E_1 - O_1 &= \sum_{i=1}^{n} x_{ii}D(1\cdots\hat{i}\cdots n|1\cdots\hat{i}\cdots n) - \sum_{i=2}^{n}\{x_{ii}D(1\cdots\hat{i}\cdots n|1\cdots\hat{i}\cdots n) \\ &\quad + q^{-1}x_{1i}D_l(i2\cdots\hat{i}\cdots n|1,2\cdots\hat{i}\cdots n)\} \\ &= x_{11}D(2\cdots n|2\cdots n) + \sum_{i=2}^{n}(-q_{i1})x_{1i}D_l(i2\cdots\hat{i}\cdots n|1,2\cdots\hat{i}\cdots n) \\ &= D(1\cdots n|1\cdots n) = D. \end{aligned}$$

(2)

$$\begin{aligned} P &= P_r(2\cdots n,1|2\cdots n,1) \\ &= P_r(2\cdots n|2\cdots n)x_{11} + \sum_{i=2}^{n} q^{-1}P_r(2\cdots\hat{i}\cdots n,1|2\cdots\hat{i}\cdots n,i)x_{i1} \\ &= O_{n-1}. \end{aligned}$$

(3) Follows by direct and tedious computations. Details are left to the reader. □

We are now ready to prove Muir's formula for the quantum generic matrix X.

Theorem 3.2. *Let $X=(x_{ij})_{n\times n}$ be the generic matrix of $K[GL_q(n)]$. The following equation holds:*

$$\sum_{m=0}^{n}(-1)^m \sum_{(i_s)\in\Phi_m} P(i_s|i_s)D(1\cdots(\hat{i_s})\cdots n|1\cdots(\hat{i_s})\cdots n)=0.$$

Proof. By Lemma 3.1(3), $E_m=O_m+O_{m-1}$ for $m=2,\dots,n-1$. Hence

$$\sum_{m=2}^{n-1}(-1)^m E_m=O_1+(-1)^{n-1}O_{n-1}=E_1-D+(-1)^{n-1}P.$$

Therefore $D+\sum_{m=1}^{n-1}(-1)^m E_m+(-1)^n P=0$, or equivalently, $\sum_{m=0}^{n}(-1)^m E_m=0$. □

In general, $P(i_s|i_s)$ and $D(1\cdots(\hat{i_s})\cdots n|1\cdots(\hat{i_s})\cdots n)$ do not commute with each other since $K[GL_q(n)]$ is not a commutative ring. However we can similarly prove the following Muir's formula.

$$\sum_{m=0}^{n}(-1)^m \sum_{(i_s)\in\Phi_m} D(1\cdots(\hat{i_s})\cdots n|1\cdots(\hat{i_s})\cdots n)P(i_s|i_s)=0.$$

4. Some remarks

Remark 4.1. In general, $X^{n-l}\,Tr_l\neq Tr_l\,X^{n-l}$ and

$$X^n-Tr_1X^{n-1}+\cdots+(-1)^{n-1}Tr_{n-1}\cdot X+(-1)^nTr_n\neq 0.$$

Similarly, ${}_qX^{n-l}\,tr^l\neq tr_q^l\,X^{n-l}$ and

$${}_qX^n-tr_{1q}\,X^{n-1}+\cdots+(-1)^{n-1}\,tr_{n-1}X+DI_{n\times n}\neq 0.$$

But the following two equations hold:

$$(X^\tau)^n-Tr_1(X^\tau)^{n-1}+\cdots+(-1)^{n-1}\,Tr_{n-1}X^\tau+(-1)^nTr_n=0,$$

$$(X^\tau)_q^n-tr_1(X^\tau)_q^{n-1}+\cdots+(-1)^{n-1}tr_{n-1}X^\tau+DI_{n\times n}=0.$$

Here X^τ is the transpose matrix of X and $(X^\tau)_q^l$ is another kind of lth q-power of X^τ.

Remark 4.2. All theorems in this paper hold for the multiparameter quantization of $GL(n)$ (for a definition see [2]). The left and right quantum minors as well as the left and right sub-permanents are defined in a similar way. Both quantum versions of the Cayley–Hamilton theorem and Muir's formula hold for the generic matrix $X=(x_{ij})_{n\times n}$ of the multiparameter quantization of $GL(n)$.

Remark 4.3. To prove the Cayley–Hamilton theorem and Muir's formula, only half of the relations are needed. For example, we consider the algebra $K\langle x_{ij}\rangle/(r_q)$, where

(r_q) is the relation ideal generated by the following relations:

$$x_{it}x_{is} - qx_{is}x_{it} = 0 \quad \text{for all } s < t \text{ and } i,$$

$$x_{it}x_{js} + qx_{jt}x_{is} - qx_{is}x_{jt} - q^2 x_{js}x_{it} = 0 \quad \text{for all } i < j \text{ and } s < t.$$

These relations span a subspace of R_q with dimension $\frac{1}{4}n^2(n^2 - 1)$, where R_q is the relation vector space for the quantum semigroup $K[M_q(n)]$. Recall that $\dim R_q = \frac{1}{2}n^2(n^2 - 1)$. Both quantum versions of the Cayley–Hamilton theorem and Muir's formula hold for the generic matrix X of $K\langle x_{ij}\rangle/(r_q)$.

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